Conditioned stable Lévy processes and Lamperti representation.

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Abstract

By killing a stable Lévy process when it leaves the positive half-line, or by conditioning it to stay positive, or by conditioning it to hit 0 continuously, we obtain three different positive self-similar Markov processes which illustrate the three classes described by Lamperti [10]. For each of these processes, we compute explicitly the infinitesimal generator from which we deduce the characteristics of the underlying Lévy process in the Lamperti representation. The proof of this result bears on the behaviour at time 0 of stable Lévy processes before their first passage time across level 0 which we describe here. As an application, we give the law of the minimum before an independent exponential time of a certain class of Lévy processes. It provides the explicit form of the spacial Wiener-Hopf factor at a particular point and the value of the ruin probability for this class of Lévy processes.

Key words and phrases: Positive self-similar Markov processes, Lamperti representation, infinitesimal generator, stable Lévy processes conditioning to stay positive, stable Lévy processes conditioning to hit 0 continuously.

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1 Introduction and preliminary results

The stochastic processes which are considered in this work take their values in the Skohorod's space $\mathcal{D}$ of càdlàg trajectories. We define this set as follows: $\Delta := +\infty$ being the cemetery point, a function $\omega : [0, \infty) \to \mathbb{R} \cup \Delta$ belongs to $\mathcal{D}$ if and only if:

– For all $t \geq \zeta(\omega)$, $\omega_t = \Delta$, where $\zeta(\omega) := \inf\{t : \omega_t = \Delta\}$ is the lifetime of $\omega \in \mathcal{D}$ and $\inf \emptyset = +\infty$.

– For all $t \geq 0$, $\lim_{s \downarrow t} \omega_s = \omega_t$ and for all $t \in (0, \zeta(\omega))$, $\lim_{s \uparrow t} \omega_s := w_{t-}$ is a finite real value.

The space $\mathcal{D}$ is endowed with the Skohorod's J_1 topology. We denote by $X : \mathcal{D} \to \mathcal{D}$ the canonical process of the coordinates and by $(\mathcal{F}_t)$ the natural Borel filtration generated by X, i.e. $\mathcal{F}_t = \sigma(X_s,\, s \leq t)$. A probability measure P_x on $\mathcal{D}$ is the law of a Lévy process if (X, P_x) starts from x, i.e. $P_x(X_0 = x) = 1$ and has independent and homogeneous increments. Note that $(X, P_x) = (x + X, P_0)$ and that the lifetime of (X, P_x) is either a.s. infinite or a.s. finite. It is well know that for any Lévy process (X, P_x) with finite lifetime $\zeta(X)$, there exists a Lévy process (X', P_x) with infinite lifetime, such that under P_x the random variable $\zeta(X)$ is exponentially distributed and independent of X' and P_x-a.s., $X_t = X'_t$, if $t < \zeta(X)$. Furthermore the parameter of the law of of $\zeta(X)$ under P_x does not depend on x.

An $\mathbb{R}_+$-valued self-similar Markov process $(X, \mathbb{P}_x)$, $x > 0$ is a strong Markov process with values in the space $\mathcal{D}$, which fulfills a scaling property, i.e. there exists a constant $\alpha > 0$ such that:

$$\textit{The law of } \ (kX_{k^{-\alpha}t},\, t \geq 0) \ \textit{under } \mathbb{P}_x \ \textit{is } \mathbb{P}_{kx}. \tag{1.1}$$

We will call these processes pssMp for short. They are much involved in many areas of probability theory. For instance, the continuous state branching process obtained as the weak limit of a re-scaled discrete branching process is a pssMp which is associated to a self-similar Lévy tree, see [8]. These processes also appear in fragmentation theory ; the mass process of a self-similar fragmentation process is itself a pssMp, [2]. The pssMp that we are going to study here have recently been obtained in [5] as limits of re-scaled random walks whose law are in the domain of attraction of a stable law, after they are conditioned to stay positive or conditioned to hit 0 at a finite time, see sections 3.2 and 3.3 below.

According to Lamperti [10], the set of pssMp splits into three exhaustive classes which can be distinguished from each other by comparing their values at their first hitting time of 0, i.e.:

$$S = \inf\{t > 0 : X_t = 0\}.$$

This classification may be summerized as follows:

- $\mathcal{C}_1$ is the class of pssMp such that $S = +\infty$, $\mathbb{P}_x$-a.s. for all starting points $x > 0$.

- $\mathcal{C}_2$ is the class of those for which $S < +\infty$ and $X_{S-} = 0$, $\mathbb{P}_x$-a.s. for all starting points $x > 0$. Processes of this class hit the level 0 in a continuous way.
- $\mathcal{C}_3$ is that of those for which $S < +\infty$ and $X_{S-} > 0$, $\mathbb{P}_x$-a.s. for all starting points $x > 0$. In that case, the process hits 0 by a negative jump.

The main result of [10] asserts that any pssMp up to its first hitting time of 0 may be expressed as the exponential of a Lévy process, time changed by the inverse of its exponential functional. Then the underlying Lévy process in the so-called Lamperti representation of $(X, \mathbb{P}_x)$ fulfils specific features depending on the class to which $(X, \mathbb{P}_x)$ belongs. More formally, let $(X, \mathbb{P}_x)$ be a pssMp starting from $x > 0$, and write the canonical process X in the following form:

$$X_t = x \exp \xi_{\tau(tx^{-\alpha})}, \quad 0 \le t < S\,, \tag{1.2}$$

where for $t < S$, $\tau(t) = \inf\{s \ge 0 : \int_0^s \exp \alpha\xi_u du \ge t\}$. Then under $\mathbb{P}_x$, $\xi = (\xi_t,\ t \ge 0)$ is a Lévy process started from 0 which law does not depend on $x > 0$ and such that

- if $(X, \mathbb{P}_x) \in \mathcal{C}_1$, then $\zeta(\xi) = +\infty$ and $\limsup_{t\to+\infty} \xi_t = +\infty$, $\mathbb{P}_x$-a.s.
- if $(X, \mathbb{P}_x) \in \mathcal{C}_2$, then $\zeta(\xi) = +\infty$ and $\lim_{t\to\infty} \xi_t = -\infty$, $\mathbb{P}_x$-a.s.
- if $(X, \mathbb{P}_x) \in \mathcal{C}_2$, then $\zeta(\xi) < +\infty$, $\mathbb{P}_x$-a.s.

Note that for any $t < \int_0^\infty \exp(\alpha\xi_s)\, ds$,

$$\tau(t) = \int_0^{x^\alpha t} \frac{ds}{X_s^\alpha}\,, \quad \mathbb{P}_x \text{ −a.s.}$$

so that (1.2) is revertible and yields a one to one relation between the class of pssMp's killed at time S and the one of Lévy processes.

Now we recall another important result of Lamperti [10] which gives the explicit form of the generator of any pssMp in terms of this of its underlying Lévy process. Let $(X, \mathbb{P}_x)$ and ξ be any such processes related as in (1.2). We will denote by $\mathcal{K}$ and $\mathcal{L}$ their respective generators and by $\mathfrak{D}_\mathcal{K}$ and $\mathfrak{D}_\mathcal{L}$ the domains of $\mathcal{K}$ and $\mathcal{L}$. Then recall that $\mathfrak{D}_\mathcal{L}$ contains all the functions with continuous second derivatives in $[-\infty, +\infty]$ and if $\tilde{f}$ is such a function then $\mathcal{L}$ is of the form:

$$\mathcal{L}\tilde{f}(x) = a\tilde{f}'(x) + \frac{\sigma}{2}\tilde{f}''(x) + \int_\mathbb{R} [\tilde{f}(x+y) - \tilde{f}(x) - \tilde{f}'(x)l(y)]\Pi(dy) - k\tilde{f}(x)\,, \tag{1.3}$$

for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $\sigma > 0$. The measure $\Pi(dx)$ is the Lévy measure of ξ on $\mathbb{R}$; it verifies $\Pi(\{0\}) = 0$ and $\int(1 \wedge |x|^2)\,\Pi(dx) < \infty$. The function $l(\cdot)$ is a bounded Borel function such that $l(y) \sim y$ as $y \to 0$. The last term $k \ge 0$ corresponds to the killing rate of ξ, that is the parameter of $\zeta(\xi)$, ($k = 0$ if ξ has infinite lifetime). *It is important to note that in the expression (1.3), the choice of the function $l(\cdot)$ is arbitrary and the coefficient a is the only one which depends on this choice.*

Theorem 6.1 of Lamperti [10] may be stated as follows:

Theorem 1 (Lamperti [10]). *If $f : [0, +\infty] \to \mathbb{R}$ is such that $f, xf', x^2 f''$ are continuous in $[0, +\infty]$, then they belong to the domain $\mathfrak{D}_{\mathcal{K}}$ of the infinitesimal generator of $(X, \mathbb{P}_x)$ which has the form*

$$
\begin{aligned}
\mathcal{K}f(x) &= \frac{1}{x^{\alpha}} \int_{\mathbb{R}^+} [f(ux) - f(x) - f'(x) l(\log u)] \Theta(du) \\
&\qquad\qquad ax^{1-\alpha} f'(x) + \frac{\sigma}{2} x^{2-\alpha} f''(x) - kx^{-\alpha} f(x),
\end{aligned}
$$

for $x > 0$, where $\Theta(du) = \Pi(du) \circ \log u$, for $u > 0$. This expression determines the law of the process $(X_t,\, 0 \le t \le S)$ under $\mathbb{P}_x$.

To present the results of this paper, let us first consider two examples in the continuous case. The first one is when $(X, \mathbb{P}_x)$ is the standard real Brownian motion absorbed at level 0. The process $(X, \mathbb{P}_x)$ is a pssMp which belongs to the class $\mathcal{C}_2$, with index $\alpha = 2$ and it is well known (see for instance [6]) that its associated Lévy process in the Lamperti representation (1.2) is given by $\xi = (B_t - t/2,\, t \ge 0)$, where B is a standard Brownian motion. The second example is when $(X, \mathbb{P}_x)$ is the Brownian motion conditioned to stay positive. This process corresponds to the three dimensional Bessel process, i.e. the norm of a three dimensional Brownian motion. Then, $(X, \mathbb{P}_x)$ is a pssMp which belongs to the class $\mathcal{C}_1$, with index $\alpha = 2$ and the underlying Lévy process is given by $\xi = (B_t + t/2,\, t \ge 0)$.

Similarly, it is possible to obtain pssMp's from any stable Lévy process (X, P_x) with index $\alpha \in (0, 2)$, throughout the same operations. More precisely, by killing (X, P_x) when it enters into the negative halfline, i.e.

$$X_t \mathbb{I}_{\{t<T\}}\,, \quad \text{with} \;\; T = \inf\{t \ge 0 : X_t \le 0\}\,,$$

one obtains a pssMp $(X, \mathbb{P}_x)$ which belongs to the class $\mathcal{C}_2$ or $\mathcal{C}_3$ according as (X, P_x) has negative jumps or not. Also by conditioning a stable Lévy process to stay positive, i.e.

$$\mathbb{P}_x = \lim_{t \to +\infty} P_x(\,\cdot \,|\, T > t)\,, \quad x > 0\,,$$

one obtains a pssMp $(X, \mathbb{P}_x)$ belonging to $\mathcal{C}_1$. One may also give a sense to the conditioning to hit 0 continuously; such processes belong to $\mathcal{C}_3$. The main goal of this paper, is to identify the underlying Lévy process in the Lamperti representation for each of these processes by computing their infinitesimal generators and using Lamperti's result recalled above. This will be done in section 3. In section 4, we deduce from the results of section 3.1, the law of the minimum before an independent exponential time for an important class of Lévy processes. It gives the expression of the Wiener-Hopf factor of these Lévy processes at a particular point, i.e. the law of $\inf_{s \le \mathbf{e}(k)} \xi_s$, where ξ is a Lévy process which characteristics are described in Corollary 1 and $\mathbf{e}(k)$ is an independent random variable with a special parameter k. We also find the law of the overall minimum for another class of Lévy processes whose law is given by Corollary 2. This calculation is equivalent to the problem of finding the

explicit form of the corresponding ruin probability which has recently been studied for other classes of Lévy processes by Lewis and Mordecki [11]. The next section is devoted to further preliminary results, the main of which having some interest in its own, independently of the rest of the paper. It extends a result of Bingham [3] and Rivero [12] which describes the asymptotic behaviour as t goes to 0 of $P_x(T \le t)$, that is the small tail of first passage times of stable Lévy processes.

2 Small tail of first passage times of stable Lévy processes

In all the sequel of this paper, (X, P_x) will be a stable Lévy process with index $\alpha \in (0,2)$, starting at $x \in \mathbb{R}$. Since stable Lévy processes have infinite lifetime, the characteristic exponent of (X, P_x) is defined by $E_0[\exp(i\lambda X_t)] = \exp[t\psi(\lambda)]$, $t \ge 0$, $\lambda \in \mathbb{R}$, where

$$\psi(\lambda) = ia\lambda + \int_{\mathbb{R}} (e^{i\lambda y} - 1 - i\lambda y \mathbb{1}_{\{|y|<1\}})\, \nu(y)\, dy\,. \tag{2.1}$$

The density of the Lévy measure is

$$\nu(y) = c_+ y^{-\alpha-1}\mathbf{1}_{\{y>0\}} + c_-|y|^{-\alpha-1}\mathbf{1}_{\{y<0\}}\,, \tag{2.2}$$

where c_+ and c_- are two nonnegative constants such that $c_+ + c_- > 0$. Note also that the constant a is related to c_+, c_- and α as follows: $a = \frac{c_+ - c_-}{1-\alpha}$, $\alpha \ne 1$. In the case where $\alpha = 1$, the process (X, P_x) will be supposed to be a symmetric Cauchy process, so we have $c_+ = c_-$ and $a = 0$. We suppose moreover that neither (X, P_x) nor $(-X, P_x)$ is a subordinator.

The main result of this section concerns the asymptotic behaviour as $t \downarrow 0$ of

$$E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t \in (0,\infty)\}})\,, \quad \text{with} \quad T = \inf\{t : X_t \le 0\}\,,$$

where f is a bounded and continuous function. This result will be used to compute the infinitesimal generator of the killed stable Lévy processes. We denote by $P_x(\,\cdot\,|\, X_t = y)$ a regular version of the law of the bridge of the Lévy process (X, P_x) from x to y, with length t. Let $p_s(z)$, $s \ge 0$, $z \in \mathbb{R}$ be the density of the semigroup of (X, P_x), then for all $s \in [0, t)$, this law is defined on $\mathcal{F}_s$ by

$$P_x(A \,|\, X_t = y) = E_x\left(\mathbb{1}_A \frac{p_{t-s}(y - X_s)}{p_t(y-x)}\right)\,, \quad A \in \mathcal{F}_s\,. \tag{2.3}$$

See [9] for a complete account on bridges of Markov processes.

Now let us recall some classical properties of densities of stable laws which may be found in [14] or [13], Chap. 3.14. When the corresponding Lévy measure is not concentrated on either $(-\infty, 0]$ or $[0,\infty)$, there are constants $C_1, C_2 > 0$, such that

$$p_1(z) \sim C_1|z|^{-\alpha-1}\,, \quad \text{as } z \to -\infty \quad \text{and} \quad p_1(z) \sim C_2 z^{-\alpha-1}\,, \quad \text{as } z \to +\infty. \tag{2.4}$$

If the Lévy measure is concentrated on $(-\infty, 0]$, then there are constants $C_3, C_4 > 0$ such that

$$p_1(z) \sim C_3|z|^{-\alpha-1}\,, \quad \text{as } z \to -\infty \quad \text{and} \quad p_1(z) \sim C_4 x^{2-\alpha} e^{-x}\,, \quad \text{as } z \to +\infty, \tag{2.5}$$

where $x = (\alpha-1)(z/\alpha)^{\alpha/(\alpha-1)}$. Note that in this second case, we have necessarily $1 < \alpha < 2$ since we have implicitly excluded subordinators of our study.

Our first lemma expresses the intuitive fact that the amplitude of a bridge from x to y of (X, P_x) tends to $|y-x|$ as its length goes to 0. It might be established more directly from a suitable estimation of the join law of $(X_t, \underline{X}_t)$, under P_x, where $\underline{X}_t := \inf_{s \le t} X_s$, however we have not found any such result in the literature.

Lemma 1. *For all $x, y > 0$,*

$$\lim_{t\to 0} P_x(T \le t \,|\, X_t = y) = 0\,.$$

Proof. First let $t > 0$ and decompose the term of the statement as

$$P_x(T \le t \,|\, X_t = y) = P_x(T \le t/2 \,|\, X_t = y) + P_x(T \in (t/2, t] \,|\, X_t = y)\,. \tag{2.6}$$

To prove the result, it is enough to show that the first term in (2.6) converges to 0 as t tends to 0. Indeed, let $(X, \hat{P}_x) := (-X, P_x)$ be the dual Lévy process, then the following identity in law between the bridge and its time reversed version is well known, see [9] for instance:

$$\big((X_{(t-s)-},\, 0 \le s \le t), P_x(\,\cdot \,|\, X_t = y)\big) = \big((X_s,\, 0 \le s \le t), \hat{P}_y(\,\cdot \,|\, X_t = x)\big)\,. \tag{2.7}$$

(We have set $X_{0-} = 0$.) Then we observe the inequality:

$$P_x(T \in (t/2, t] \,|\, X_t = y) \le \hat{P}_y(T \le t/2 \,|\, X_t = x)\,.$$

If the first term of (2.6) converges to 0 in any case, then by applying the result to the bridge of the dual process and the above inequality, we show that the second term of (2.6) converges also to 0.

Now, let us prove that the first term of (2.6) converges to 0 as t goes to 0. Recall that $\underline{X}_t := \inf_{s \le t} X_s$. From (2.3) the first term is

$$\begin{aligned} P_x(T \le t/2 \,|\, X_t = y) &= E_x\left(\mathbb{1}_{\{T \le t/2\}} \frac{p_{t/2}(y - X_{t/2})}{p_t(y-x)}\right) \\ &= E_0\left(\mathbb{1}_{\{\underline{X}_{t/2} \le -x\}} \frac{p_1(2^{1/\alpha} t^{-1/\alpha}[y - x - X_{t/2}])}{2^{-1/\alpha} p_1(t^{-1/\alpha}[y-x])}\right)\,, \end{aligned} \tag{2.8}$$

where the second identity follows from the fact that $p_t(z) = t^{-1/\alpha} p_1(t^{-1/\alpha} z)$, for all $t > 0$.

From classical properties of stable Lévy processes, we have $P_0(\underline{X}_{t/2} \le -x) \to 0$ as $t \to 0$ and $p_1(0) > 0$. Therefore, if $x = y$, then since $z \mapsto p_1(z)$ is bounded on $\mathbb{R}$,

we see that the right hand side of (2.8) tends to 0 as t goes to 0. So the lemma is proved when $x = y$.

Set $q = 1 - 2^{-1/(2\alpha)}$ and suppose that $y > x$, then again we develop the right hand side of (2.8) as the sum:

$$E_0\left(\mathbb{I}_{\{\underline{X}_{t/2}\le -x,\, X_{t/2}\le q(y-x)\}}\frac{p_1(2^{1/\alpha}t^{-1/\alpha}[y-x-X_{t/2}])}{2^{-1/\alpha}p_1(t^{-1/\alpha}[y-x])}\right)$$
$$+E_0\left(\mathbb{I}_{\{\underline{X}_{t/2}\le -x,\, X_{t/2}\ge q(y-x)\}}\frac{p_1(2^{1/\alpha}t^{-1/\alpha}[y-x-X_{t/2}])}{2^{-1/\alpha}p_1(t^{-1/\alpha}[y-x])}\right). \quad (2.9)$$

Note that on the event $\{X_{t/2} \le q(y-x)\}$, we have

$$2^{1/\alpha}t^{-1/\alpha}[y-x-X_{t/2}] \ge t^{-1/\alpha}[y-x] > 0.$$

So, from (2.4) and (2.5), there is a time t_1 and a finite constant c_1 (both non random) such that for all $0 < t \le t_1$, on the event $\{X_{t/2} \le q(y-x)\}$ we have

$$\frac{p_1(2^{1/\alpha}t^{-1/\alpha}[y-x-X_{t/2}])}{p_1(t^{-1/\alpha}[y-x])} \le c_1\,. \quad (2.10)$$

Hence from Lebesgue theorem of dominated convergence the first term in (2.9) tends to 0 at t goes to 0. Now call $\hat{p}_t(z) := p_t(-z)$ the semigroup of the dual process $(X, \hat{P}_0)$. Since bridges of Lévy processes have no fixed discontinuities, see [9], from (2.7), the second term in (2.9) may be written as

$$\begin{aligned}
&E_0\left(\mathbb{I}_{\{\underline{X}_{t/2}\le -x,\, X_{t/2}\ge q(y-x)\}}\frac{p_1(2^{1/\alpha}t^{-1/\alpha}[y-x-X_{t/2}])}{2^{-1/\alpha}p_1(t^{-1/\alpha}[y-x])}\right)\\
=\;& P_x(\underline{X}_{t/2}\le 0,\, X_{t/2}\ge q(y-x)+x \,|\, X_t = y)\\
=\;& \hat{P}_y(\underline{X}_{t/2}\le 0,\, X_{t/2}\ge q(y-x)+x \,|\, X_t = x)\\
=\;& \hat{E}_0\left(\mathbb{I}_{\{\underline{X}_{t/2}\le 0,\, X_{t/2}\ge (1-q)(x-y)\}}\frac{\hat{p}_1(2^{1/\alpha}t^{-1/\alpha}[x-y-X_{t/2}])}{2^{-1/\alpha}\hat{p}_1(t^{-1/\alpha}[x-y])}\right).
\end{aligned}$$

On the event $\{X_{t/2} \ge (1-q)(x-y)\}$, we have

$$2^{1/\alpha}t^{-1/\alpha}[x-y-X_{t/2}] \le (2^{1/\alpha} - 2^{1/(2\alpha)})t^{-1/\alpha}[x-y] < 0.$$

If (X, P_x) has positive jumps, then from (2.5), $\hat{p}_1(z) \sim C_3|z|^{-\alpha-1}$ as $z \to -\infty$, thus there is a time t_2 and a finite constant c_2 (both non random) such that for all $0 < t \le t_2$, on the event $\{X_{t/2} \ge (1-q)(x-y)\}$ we have

$$\frac{\hat{p}_1(2^{1/\alpha}t^{-1/\alpha}[x-y-X_{t/2}])}{\hat{p}_1(t^{-1/\alpha}[x-y])} \le c_2\,, \quad (2.11)$$

hence, again the second term in (2.9) tends to 0 at t goes to 0.

So we have proved the lemma when $y > x$ and (X, P_x) has positive jumps. By a time reversal argument, it is easy to see that the same result holds when $y < x$ and (X, P_x) has negative jumps. It remains to show the result when $y < x$ and (X, P_x) has no negative jumps. In this case, put $T_y = \inf\{t : X_t = y\}$, then from the Markov property applied at time T_y, we have

$$P_x(T \leq t \,|\, X_t = y) = \int_0^t P_x(T_y \in ds \,|\, X_t = y) P_y(T \leq t - s \,|\, X_{t-s} = y) \,. \qquad (2.12)$$

But we already proved above that $P_y(T \leq t - s \,|\, X_{t-s} = y)$ tends to 0 as $t - s$ goes to 0. This ends the proof of the lemma. □

Recall that the characteristic exponent of $(X, \mathbb{P})$ may also be written in the following form for $\alpha \in (0,1) \cup (1,2)$,

$$E_0[\exp(i\lambda X_t)] = \exp[-ct|\lambda|^\alpha(1 - i\beta \mathrm{sgn}(\lambda) \tan(\pi\alpha/2))] \,, \quad \lambda \in \mathbb{R} \,, \qquad (2.13)$$

where

$$c = (c_+ + c_-)\Gamma(-\alpha) \cos\frac{\pi\alpha}{2} \quad \text{and} \quad \beta = (c_+ - c_-)/(c_+ + c_-) \,,$$

see for instance Sato [13], Theorem 14.10 and its proof p.83–85. It has been proved by Bingham [3], Proposition 3.b and Theorem 4.b, and Rivero [12], section 2.3 that

$$\lim_{t \downarrow 0} \frac{1}{t} P_x(T \leq t) = \frac{k}{x^\alpha} \,, \qquad (2.14)$$

where the constant k is explicitly computed in [3] and is given by:

$$k = c(1 + \beta^2 \tan^2(\pi\alpha/2))^{1/2} \Gamma(\alpha) \sin(\pi\alpha\rho)/\pi \,. \qquad (2.15)$$

By definition, $\rho := P_0(X_1 < 0)$ and it is well known that this rate has the expression

$$\rho = \frac{1}{2} - (\pi\alpha)^{-1} \arctan(\beta \tan(\pi\alpha/2)) \,.$$

Note that we always have $\alpha\rho \leq 1$. Moreover, we easily check that (X, P_x) has no negative jumps if and only if one of the three following conditions holds

$$c_- = 0 \Leftrightarrow \beta = 1 \Leftrightarrow \alpha\rho = 1 \,.$$

For $\alpha = 1$, the expressions (2.13) and (2.15) are reduced to $E_0[\exp(i\lambda X_t)] = \exp(-c_+\pi t|\lambda|)$ and $k = c_+ (= c_-)$, respectively (although the value of k in this case is ambiguous in [3], it will be confirmed in section 3.2).

Then in section 3.2 we will provide another means to compute the expression of the constant k, see formula (3.12). Note also that Rivero's result [12] concerns the more general setting of positive self-similar Markov processes. Besides, in the case where (X, P_x) has no negative jumps, we have $k = 0$ but Proposition 3.b of [3] gives an explicit form of the asymptotic behaviour of $P_x(T < t)$, as $t \downarrow 0$. The next theorem completes Bingham and Rivero's result.

Theorem 2. *For all $x > 0$, and all bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$,*

$$\lim_{t\to 0} \frac{1}{t} E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t\in(0,\infty)\}}) = \frac{f(x)}{x^\alpha}\left(k - \frac{c_-}{\alpha}\right) ,$$

where c_- and k are respectively defined in (2.2) and (2.15).

Proof. Let $\delta \in (0, x)$ and define $I_{\delta,x} := [x-\delta, x+\delta]$. Let also f be a bounded and continuous function and write:

$$\begin{aligned}\frac{1}{t} E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t\in(0,\infty)\}}) &= \frac{1}{t} E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t\in I_{\delta,x}\cap(0,\infty)\}}) \\ &\quad + \frac{1}{t} E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t\in I^c_{\delta,x}\cap(0,\infty)\}}) .\end{aligned}$$

Then we express the second term as follows:

$$\begin{aligned}&\frac{1}{t} E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t\in I^c_{\delta,x}\cap(0,\infty)\}}) = \\ &\qquad \int_{I^c_{\delta,x}\cap(0,\infty)} f(y) P_x(t \ge T \mid X_t = y) \frac{p_1(t^{-1/\alpha}(y-x))}{t^{1+1/\alpha}}\, dy . \qquad (2.16)\end{aligned}$$

From (2.4) and (2.5), there is a constant $C_5 > 0$ such that for $|x|$ sufficiently large,

$$p_1(x) \le C_5 |x|^{-\alpha-1} ,$$

hence there exist $C_6 > 0$ and t_1 (which may depend on x) such that for all $y \in I^c_{\delta,x}$ and for all $0 < t \le t_1$,

$$p_1(t^{-1/\alpha}(y-x)) \le C_6 t^{1+1/\alpha} .$$

Therefore, from the Lebesgue theorem of dominated convergence and Lemma 1, the expression in (2.16) tends to 0 as t goes to 0.

Now recalling our first equality, we have for any $\delta \in (0, x)$,

$$\lim_{t\to 0} \frac{1}{t} E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t\in(0,\infty)\}}) = \lim_{t\to 0} \frac{1}{t} E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t\in I_{\delta,x}\cap(0,\infty)\}}) . \qquad (2.17)$$

Set $b^-_{\delta,x} = \inf\{f(y),\, y \in I_{\delta,x}\}$ and $b^+_{\delta,x} = \sup\{f(y),\, y \in I_{\delta,x}\}$. From our hypothesis on f, $b^-_{\delta,x}$ and $b^+_{\delta,x}$ are finite and from the equality above, we have

$$\begin{aligned}&b^-_{\delta,x} \lim_{t\to 0} \frac{1}{t} [P_x(T \le t,\, X_t \in (0,\infty)) - P_x(T \le t, X_t \in I^c_{\delta,x} \cap (0,\infty))] \le \\ &\quad \lim_{t\to 0} \frac{1}{t} E_x(f(X_t)\mathbb{1}_{\{T\le t,\, X_t\in(0,\infty)\}}) \le \lim_{t\to 0} b^+_{\delta,x} \frac{1}{t} P_x(T \le t,\, X_t \in (0,\infty)). \qquad (2.18)\end{aligned}$$

But applying again (2.17) with $f \equiv 1$, we find $\lim_{t\to 0} \frac{1}{t} P_x(T \le t, X_t \in I^c_{\delta,x} \cap (0,\infty)) = 0$.

Now write:

$$\frac{1}{t}P_x(T \le t,\, X_t \in (0,\infty)) = \frac{1}{t}P_x(T \le t) - \frac{1}{t}P_x(X_t \in (-\infty, 0])\,,$$

and apply (2.14) together with the fact that $\lim_{t\to 0} \frac{1}{t}P_x(X_t \in (-\infty,0]) = c_-/(\alpha x^\alpha)$, see for instance [1], Exercise I.1.

Finally note that δ is arbitrarily small in the inequalities (2.18) and since f is continuous, $b^-_{\delta,x}$ and $b^+_{\delta,x}$ tend to $f(x)$ as δ goes to 0. This allows us to conclude. □

3 Killed or conditioned stable processes as pssMp

In this section, we compute the characteristics of the underlying Lévy process in the Lamperti representation of a pssMp $(X,\mathbb{P}_x)$ when this process is either a stable Lévy process which is killed when it first hits the positive half-line (section 3.1) or a stable Lévy process conditioned to stay positive (section 3.2) or a stable Lévy process conditioned to hit 0 continuously (section 3.3). If (X, P_x) is a stable subordinator, then it can be considered as its own version conditioned to stay positive and in this case and the characteristics of the underlying Lévy process have been computed by Lamperti [10], Section 6. Except in this situation, the cases where (X,P_x) or $(-X,P_x)$ is a subordinator have no interest in this study, so they will be implicitly excluded in the sequel. Also, as already mentioned in the introduction, since all our study is well known when (X,P_x) is the standard Brownian motion, we will always suppose that $\alpha \neq 2$.

3.1 The killed process

In this subsection, we suppose that $(X,\mathbb{P}_x)$, $x>0$ is a stable Lévy process with index $\alpha \in (0,2)$ which is killed when it first leaves the positive half-line. To define this process more formally, let (X,P_x) be a stable Lévy process starting at $x>0$. *We keep the same notations as in Section 2 for the characteristics of* (X,P_x). Recall that $T = \inf\{t \ge 0 : X_t \le 0\}$, then the probability measure $\mathbb{P}_x$ is the law under P_x of the process

$$X_t 1\!\!1_{\{t<T\}}\,, \quad t \ge 0\,. \tag{3.1}$$

(Note that rather than the *killed* process, we could also call $(X,\mathbb{P}_x)$ the initial Lévy process (X,P_x) *absorbed* at level 0). It is not difficult to see, that the process $(X,\mathbb{P}_x)$ is a positive self-similar Markov process with index α such that $S<\infty$, $\mathbb{P}_x$-a.s. Furthermore, if (X,P_x) has no negative jumps, then $(X,\mathbb{P}_x)$ ends continuously at 0, so it belongs to the class $\mathcal{C}_2$. If (X,P_x) has negative jumps, then it is known that it crosses the level 0 for the first time by a jump, so $(X,\mathbb{P}_x)$ ends by a jump at 0 and belongs to the class $\mathcal{C}_3$. We will compute the infinitesimal generator of $(X,\mathbb{P}_x)$ and deduce from its expression the law of the underlying Lévy process ξ associated to $(X,\mathbb{P}_x)$ in the Lamperti representation.

Specializing the expression given in the introduction for stable Lévy processes, we obtain the infinitesimal generator $\mathcal{A}$ with domain $\mathfrak{D}_\mathcal{A}$ of the process (X, P_x):

$$\mathcal{A}f(x) = af'(x) + \int_\mathbb{R} (f(x+y) - f(x) - yf'(x)\mathbb{1}_{\{|y|<1\}})\nu(y)dy \tag{3.2}$$

for $f \in \mathfrak{D}_\mathcal{A}$, where we recall from the beginning of section 2 that $\nu(y) = c_+ y^{-\alpha-1}\mathbf{1}_{\{y>0\}} + c_-|y|^{-\alpha-1}\mathbf{1}_{\{y<0\}}$ is the density of the the Lévy measure and that $c_- \geq 0$, $c_+ \geq 0$, $a = \frac{c_+ - c_-}{1-\alpha}$, if $\alpha \neq 1$ and $a = 0$, $c_+ = c_-$, if $\alpha = 1$.

In the sequel, we will denote by $\mathcal{K}$ the infinitesimal generator of the killed process $(X, \mathbb{P}_x)$. Note that since the state space of this process is $[0, \infty)$ and 0 is an absorbing state, the domain of $\mathcal{K}$, that will be denoted by $\mathfrak{D}(\mathcal{K})$, is included in the set $\{f : [0, \infty) \to \mathbb{R} : f(0) = 0\}$. From the expression of the infinitesimal generator $\mathcal{A}$, we can deduce this of $\mathcal{K}$ as shows the following result.

Theorem 3. *Let $(X, \mathbb{P}_x)$ be the pssMp which is defined in (3.1) and let $\mathcal{K}$ be its generator. Let $f \in \mathfrak{D}_\mathcal{K}$ such that the function $\tilde{f}$ defined on $\mathbb{R}$ by*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

belongs to $\mathfrak{D}_\mathcal{A}$, then

$$\mathcal{K}f(x) = \mathcal{A}\tilde{f}(x) - \frac{f(x)}{x^\alpha}\left(k - \frac{c-}{\alpha}\right), \quad x > 0\,, \quad \mathcal{K}f(0) = 0\,,$$

where the constant k is defined in Lemma 2. The generator $\mathcal{K}$ can also be written as

$$\begin{aligned} \mathcal{K}f(x) &= \int_{\mathbb{R}^+} \frac{1}{x^\alpha}(f(ux) - f(x) - xf'(x)(u-1)\mathbb{1}_{\{|u-1|<1\}})\nu(u-1)du \\ &\qquad + ax^{1-\alpha}f'(x) - kx^{-\alpha}f(x)\,. \end{aligned}$$

Remark: We emphasize that the set of functions which is used in the above statement to describe the generator $\mathcal{K}$ contains at least all functions of the set $\{f : [0, \infty) \to \mathbb{R} : f(0) = 0\}$ such that $\tilde{f} \in \mathcal{C}_b^2(\mathbb{R})$.

Proof. Recall that $T = \inf\{t \geq 0 : X_t \leq 0\}$, $S = \inf\{t \geq 0 : X_t = 0\}$ and let f be a function which is as in the statement of the theorem. Then note that

$$\begin{aligned} \mathbb{E}_x(f(X_t)) &= \mathbb{E}_x(f(X_t)\mathbf{1}_{\{t<S\}} + f(0)\mathbb{1}_{\{t\geq S\}}) = E_x(\tilde{f}(X_t)\mathbf{1}_{\{t<T\}}) \\ &= E_x(\tilde{f}(X_t)) - E_x(\tilde{f}(X_t)\mathbf{1}_{\{T\leq t\}})\,. \end{aligned}$$

So, for any $x > 0$ the generator of the killed process $(X, \mathbb{P}_x)$ is given by:

$$\begin{aligned} \mathcal{K}f(x) &= \lim_{t\to\infty} \frac{1}{t}\,\mathbb{E}_x(f(X_t) - f(x)) \\ &= \lim_{t\to 0} \frac{1}{t}\,[E_x(\tilde{f}(X_t)) - \tilde{f}(x)] - \lim_{t\to 0} \frac{1}{t}\,E_x(\tilde{f}(X_t)\mathbf{1}_{\{T\leq t\}}) \\ &= \mathcal{A}\tilde{f}(x) - \frac{f(x)}{x^\alpha}\left(k - \frac{c-}{\alpha}\right). \end{aligned}$$

Where the last equality comes from Lemma 2, since $\tilde{f}$ is continuous and bounded. The value of $\mathcal{K}f$ at 0 is easily computed.

To prove the second assertion of the theorem, write

$$\mathcal{K}f(x) = af'(x) - \frac{f(x)}{x^\alpha}(k - \frac{c-}{\alpha}) + \int_{\mathbb{R}} (\tilde{f}(x+y) - f(x) - yf'(x)\mathbb{1}_{\{|y|<1\}})\nu(y)\,dy$$

and let I be the integral term above. Then make the change of variable $y = x(u-1)$ to obtain,

$$I = \frac{1}{x^\alpha}\int_{u\in\mathbb{R}} [\tilde{f}(xu) - f(x) - x(u-1)f'(x)\mathbb{1}_{\{|x(u-1)|<1\}}]\nu(u-1)\,du\,.$$

We rewrite I in the following form:

$$\begin{array}{rl} I = & \frac{1}{x^\alpha}\int_{(u>0)}[\tilde{f}(xu) - f(x) - x(u-1)f'(x)\mathbb{1}_{\{|u-1|<1\}}]\nu(u-1)\,du \\ + & \frac{1}{x^\alpha}\int_{(u>0)}[x(u-1)f'(x)(\mathbb{1}_{\{|u-1|<1\}} - \mathbb{1}_{\{|x(u-1)|<1\}}]\nu(u-1)\,du \\ + & \frac{1}{x^\alpha}\int_{(u<0)}[\tilde{f}(xu) - f(x) - x(u-1)f'(x)\mathbb{1}_{\{|x(u-1)|<1\}}]\nu(u-1)\,du \end{array}$$

and we call each if these integrals I_1, I_2, I_3 respectively. Integral I_1 stays as it is but I_2 and I_3 require additional calculations:

$$I_3 = -\frac{f(x)}{x^\alpha}\int_{(u<0)} \nu(u-1)du - \frac{1}{x^\alpha}\int_{(u<0)} x(u-1)f'(x)\mathbb{1}_{\{|x(u-1)|<1\}}\nu(u-1)\,du\,.$$

Now suppose that $\alpha \neq 1$ (the case $\alpha = 1$ being much simpler). We may verify (after fastidious calculations) that the sum of I_2 and the second term of I_3 gives

$$\frac{c_+ - c_-}{1-\alpha}(1 - x^{\alpha-1})\frac{f'(x)}{x^{\alpha-1}} = af'(x)(x^{1-\alpha} - 1)\,,$$

since $a = \frac{c_+ - c_-}{1-\alpha}$. We finally calculate the first term of I_3:

$$-\frac{f(x)}{x^\alpha}\int_{(u<0)} \nu(u-1)\,du = -\frac{f(x)}{x^\alpha}\frac{c_-}{\alpha}\,.$$

Then by adding again all the different parts together, we find for the expression of $\mathcal{K}f(x)$:

$$\begin{aligned} \mathcal{K}f(x) &= af'(x) - \frac{f(x)}{x^\alpha}(k - \frac{c-}{\alpha}) + I_1 + af'(x)(x^{1-\alpha} - 1) - \frac{f(x)}{x^\alpha}\frac{c-}{\alpha} \\ &= \frac{a}{x^{\alpha-1}}f'(x) + I_1 - \frac{f(x)}{x^\alpha}k\,, \end{aligned}$$

which ends the proof. $\square$

Let ξ be the underlying Lévy process in the Lamperti representation of $(X, \mathbb{P}_x)$, as it is stated in (1.2). Recall that ξ may have finite lifetime, so its characteristic exponent Φ is defined by

$$\mathbb{E}[\exp(i\lambda\xi_t)\mathbb{I}_{\{t<\zeta(\xi)\}}] = \exp[t\Phi(\lambda)]\,, \quad \lambda \in \mathbb{R}\,. \tag{3.3}$$

Using Lamperti's result which is recalled in Theorem 1 in the introduction and Theorem 3, we may now give the explicit form of the generator of ξ in the special setting of this subsection.

Corollary 1. *Let ξ be the Lévy process in Lamperti representation (1.2) of the pssMp which is $(X, \mathbb{P}_x)$ defined in (3.1). The infinitesimal generator $\mathcal{L}$ of ξ with domain $\mathfrak{D}_{\mathcal{L}}$ is given by*

$$\mathcal{L}f(x) = af'(x) + \int_{\mathbb{R}} (f(x+y) - f(x) - f'(x)(e^y - 1)\mathbb{I}_{\{|e^y-1|<1\}})\pi(y)dy - kf(x)\,,$$

for any $f \in \mathfrak{D}_{\mathcal{L}}$ and $x \in \mathbb{R}$, where $\pi(y) = e^y\nu(e^y - 1)$, $y \in \mathbb{R}$. Equivalently, the characteristic exponent of ξ is given by

$$\Phi(\lambda) = ia\lambda + \int_{\mathbb{R}} (e^{i\lambda y} - 1 - i\lambda(e^y - 1)\mathbb{I}_{\{|e^y-1|<1\}})\pi(y)dy - k\,.$$

The process $(X, \mathbb{P}_x)$ belongs to the class $\mathcal{C}_3$ if $k > 0$ and it belongs to the class $\mathcal{C}_2$ if $k = 0$. In the first case the Lévy process ξ has finite lifetime with parameter k, in the second case, it has infinite lifetime.

It is rather unusual to see $l(y) = (e^y - 1)\mathbb{I}_{\{|e^y-1|<1\}}$ as the compensating function in the expression of the infinitesimal generator or the characteristic exponent of a Lévy process. However, as noticed in the introduction, any function l such that $l(y) \sim y$, as $y \to 0$ may be chosen and the more classical function $l(y) = y\mathbb{I}_{\{|y|<1\}}$, would have the effect of replacing the parameter a by another one which expression is rather complicated.

Let us consider the unkilled version of ξ, i.e. the Lévy process $\tilde{\xi}$ with characteristic exponent

$$\tilde{\Phi}(\lambda) = ia\lambda + \int_{\mathbb{R}} (e^{i\lambda y} - 1 - i\lambda(e^y - 1)\mathbb{I}_{\{|e^y-1|<1\}})\pi(y)dy\,.$$

A natural question is to know wether if the process $\tilde{\xi}$ oscillates, drifts to $-\infty$ or drifts to $+\infty$. Let us show that the three situations may happen depending on the relative values of c_-, c_+ and α. From the expression of $\tilde{\Phi}$, we see that $\tilde{\xi}$ is integrable and

$$\begin{aligned}\mathbb{E}(\tilde{\xi}_1) = -i\tilde{\Phi}'(0) = a + c_+ &\left(\int_0^{\log 2} \frac{(1+y-e^y)e^y}{(e^y-1)^{\alpha+1}}\,dy + \int_{\log 2}^{\infty} \frac{ye^y}{(e^y-1)^{\alpha+1}}\,dy\right) \\ &+c_- \int_{-\infty}^{0} \frac{(1+y-e^y)e^y}{(1-e^y)^{\alpha+1}}\,dy\,.\end{aligned} \tag{3.4}$$

(Here $\mathbb{P}$ can be any of the measures $\mathbb{P}_x$, $x > 0$). On the one hand, it is clear from the classification which is recalled in the introduction that when (X, P_x) has no negative jumps (i.e. $c_- = 0$), then the Lévy process $\tilde{\xi} = \xi$ drifts towards $-\infty$, so that

$$\frac{c_+}{1-\alpha} + c_+ \left(\int_0^{\log 2} \frac{(1 + y - e^y)e^y}{(e^y - 1)^{\alpha+1}}\, dy + \int_{\log 2}^{\infty} \frac{y e^y}{(e^y - 1)^{\alpha+1}}\, dy \right) < 0\,, \tag{3.5}$$

for all $c_+ > 0$ and $\alpha \in (1, 2)$. (Recall that in the spectrally one side case, we have necessarily $\alpha \in (1, 2)$.) On the other hand, when (X, P_x) has no positive jumps ($c_+ = 0$), then it is easy to derive from (3.4) that for any $c_- > 0$ fixed, $\lim_{\alpha \downarrow 1} \mathbb{E}(\tilde{\xi}_1) = +\infty$ and $\lim_{\alpha \uparrow 2} \mathbb{E}(\tilde{\xi}_1) = -\infty$. Since $\alpha \mapsto \mathbb{E}(\tilde{\xi}_1)$ is continuous, there are values of $\alpha \in (1, 2)$ for which $\tilde{\xi}$ drifts to $-\infty$, oscillates or drifts to $+\infty$. This argument and (3.5) show that for all $c_- > 0$ and $c_+ > 0$, there are values of $\alpha \in (1, 2)$ for which $\tilde{\xi}$ drifts to $-\infty$.

3.2 The process conditioned to stay positive

We consider again a stable Lévy process (X, P_x) as it is defined as in section 2. Formally, the process (X, P_x) conditioned to stay positive is an h-transform of the killed process defined in section 3.1, i.e.

$$\mathbb{P}_x^{\uparrow}(A) = h^{-1}(x) E_x(h(X_t) \mathbb{I}_A \mathbb{I}_{\{t<T\}})\,, \quad x > 0,\, t \geq 0,\, A \in \mathcal{F}_t\,, \tag{3.6}$$

where $h(x) = x^{\alpha\rho}$. The function h being positive and harmonic for the killed process, formula (3.6) defines the law of a strong homogeneous Markov process. Moreover this process is $(0, \infty)$-valued and it is clear that it inherits the scaling property with index α from (X, P_x). Hence $(X, \mathbb{P}_x^{\uparrow})$ yields an example of pssMp which belongs to the class $\mathcal{C}_1$. The following more intuitive (but no less rigorous) construction of the law $\mathbb{P}_x^{\uparrow}$ justifies that $(X, \mathbb{P}_x^{\uparrow})$ is called *the Lévy process* (X, P_x) *conditioned to stay positive*

$$\mathbb{P}_x^{\uparrow}(A) = \lim_{t \to +\infty} P_x(A \,|\, T > t)\,, \quad x > 0,\, t \geq 0,\, A \in \mathcal{F}_t\,.$$

We refer to [4] for a general account on Lévy processes conditioned to stay positive. In particular it is proved in [4] that $(X, \mathbb{P}_x^{\uparrow})$ drifts to $+\infty$ as $t \to +\infty$, i.e.

$$\mathbb{P}_x^{\uparrow}\big(\lim_{t \to +\infty} X_t = +\infty\big) = 1\,. \tag{3.7}$$

Let us also mention that this conditioning has a discrete time counterpart for random walks. Let μ be a law which is in the domain of attraction of the stable law (X_1, P_0) and let $S^{\uparrow}$ be a random walk with law μ which is conditioned to stay positive. Then the process $(X, \mathbb{P}_x^{\uparrow})$ may be obtained as the limit in law of the process $(n^{1/\alpha} S^{\uparrow}_{[nt]},\, t \geq 0)$, as n tends to ∞. This invariance principle has recently been proved in [5].

Since $(X,\mathbb{P}_x^\uparrow)$ is an h-process of the killed process $(X,\mathbb{P}_x)$ defined at the previous subsection, its infinitesimal generator, that we denote by $\mathcal{K}^\uparrow$, may be derived from $\mathcal{K}$ as follows

$$\mathcal{K}^\uparrow f(x) = \frac{1}{h(x)}\mathcal{K}(hf)(x)\,, \quad x>0\,, \;\; f\in\mathfrak{D}_{\mathcal{K}^\uparrow}\,. \tag{3.8}$$

From (3.8) and Theorem 3, we obtain for $x>0$ and $f\in\mathfrak{D}_{\mathcal{K}^\uparrow}$:

$$\begin{aligned} x^\alpha\mathcal{K}^\uparrow f(x) &= \frac{1}{x^{\alpha\rho}}\int_{\mathbb{R}^+}[(hf)(ux)-(hf)(x)-x(hf)'(x)(u-1)\mathbb{I}_{\{|u-1|<1\}}]\nu(u-1)du \\ &\qquad +ax(hf)'(x)-k(hf)(x) \\ &= \int_{\mathbb{R}^+}[u^{\alpha\rho}f(ux)-f(x)-(\alpha\rho f(x)+xf'(x))(u-1)\mathbb{I}_{\{|u-1|<1\}}]\nu(u-1)du \\ &\qquad +axf'(x)+(a\alpha\rho-k)f(x)\,. \end{aligned}$$

Let us denote by J the integral term in the above expression and define $\nu^\uparrow(u) = u^{\alpha\rho}\nu(u-1)$. Then

$$\begin{aligned} J &= \int_{\mathbb{R}^+}[f(ux)-u^{-\alpha\rho}f(x)-(\alpha\rho f(x)+xf'(x))u^{-\alpha\rho}(u-1)\mathbb{I}_{\{|u-1|<1\}}]\nu^\uparrow(u)du \\ &= \int_{\mathbb{R}^+}[f(ux)-f(x)-xf'(x)(u-1)\mathbb{I}_{\{|u-1|<1\}}]\nu^\uparrow(u)du + \\ &\quad \int_{\mathbb{R}^+}[u^{\alpha\rho}-1-\alpha\rho(u-1)\mathbb{I}_{\{|u-1|<1\}}]\nu(u-1)du\, f(x) + \\ &\quad \int_{\mathbb{R}^+}(u^{\alpha\rho}-1)(u-1)\mathbb{I}_{\{|u-1|<1\}}\nu(u-1)du\, xf'(x)\,. \end{aligned}$$

The infinitesimal generator of the process $(X,\mathbb{P}_x^\uparrow)$ is then

$$\begin{aligned} \mathcal{K}^\uparrow f(x) &= \frac{1}{x^\alpha}\int_{\mathbb{R}^+}[f(ux)-f(x)-xf'(x)(u-1)\mathbb{I}_{\{|u-1|<1\}}]\nu^\uparrow(u)du \\ &\qquad +(a+a_1)x^{1-\alpha}f'(x)+(a\alpha\rho+a_2-k)x^{-\alpha}f(x)\,, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \nu^\uparrow(u) &= u^{\alpha\rho}\nu(u-1) \\ a_1 &= c_+\int_0^1\frac{(1+x)^{\alpha\rho}-1}{x^\alpha}\,dx + c_-\int_0^1\frac{(1-x)^{\alpha\rho}-1}{x^\alpha}\,dx \qquad (3.10) \\ a_2 &= c_+\left(\int_0^1\frac{(1+x)^{\alpha\rho}-1-\alpha\rho x}{x^{\alpha+1}}\,dx+\int_1^\infty\frac{(1+x)^{\alpha\rho}-1}{x^{\alpha+1}}\,dx\right) \\ &\quad +c_-\int_0^1\frac{(1-x)^{\alpha\rho}-1+\alpha\rho x}{x^{\alpha+1}}\,dx \qquad (3.11) \end{aligned}$$

and a, k and ρ are given in section 2. Note that since $(X,\mathbb{P}_x^\uparrow)$ belongs to $\mathcal{C}_1$, the killing rate $a\alpha\rho+a_2-k$ in the expression (3.9) of its generator must be zero, which

gives the following expression for the constant k:

$$k = a\alpha\rho + a_2\,. \tag{3.12}$$

From (2.15), the value of k is explicit in terms of the constants c_+, c_- and α, so we should be able to compute the integrals in the expressions (3.10) and (3.11) of a_1 and a_2. However, the calculation of these integrals relies to special functions and its seems to be possible to check that (2.15) and (3.12) coincide only in the trivial cases $\alpha\rho = 1$ and $\alpha = 1$.

As for $(X, \mathbb{P}_x)$ in the previous subsection, we may now apply Theorem 1 together with (3.9) to compute the characteristics of the underlying Lévy process in the Lamperti representation of $(X, \mathbb{P}_x^\uparrow)$.

Corollary 2. *Let $\xi^\uparrow$ be the Lévy process in the Lamperti representation (1.2) of the pssMp $(X, \mathbb{P}_x^\uparrow)$ which is defined in (3.6). The infinitesimal generator $\mathcal{L}^\uparrow$ of $\xi^\uparrow$ with domain $\mathfrak{D}_{\mathcal{L}^\uparrow}$ is given by*

$$\mathcal{L}^\uparrow f(x) = a^\uparrow f'(x) + \int_{\mathbb{R}} (f(x+y) - f(x) - f'(x)(e^y - 1)\mathbb{1}_{\{|e^y-1|<1\}})\pi^\uparrow(y)dy\,,$$

for any $f \in \mathfrak{D}_{\mathcal{L}^\uparrow}$ and $x > 0$, where $\pi^\uparrow(y) = e^{(\alpha\rho+1)y}\nu(e^y - 1)$, $y \in \mathbb{R}$ and $a^\uparrow = a + a_1$, the constant a_1 being defined in (3.10). Equivalently, the characteristic exponent of $\xi^\uparrow$ is given by

$$\Phi^\uparrow(\lambda) = ia^\uparrow\lambda + \int_{\mathbb{R}} (e^{i\lambda y} - 1 - i\lambda(e^y - 1)\mathbb{1}_{\{|e^y-1|<1\}})\pi^\uparrow(y)dy\,.$$

It follows from (3.7) that the underlying Lévy process $\xi^\uparrow$ drifts to $+\infty$. This process being integrable, it means in particular that $0 < \mathbb{E}(\xi_1^\uparrow) = -i\Phi^\uparrow(0) < \infty$.

3.3 The process conditioned to hit 0 continuously

Let S be an integer valued random walk which law is in the domain of attraction of the stable law (X_1, P_0). For $y \in \mathbb{Z} \setminus \{0\}$, define the law of the chain $S_y^\searrow$ as this of the random walk S_y starting from y and conditioned to hit 0 as follows :

$$\begin{aligned}(S_y^\searrow(n),\, 0 \le n \le \tau^\searrow_{(-\infty,0]}) &\stackrel{(d)}{=} [(S_y(n),\, 0 \le n \le \tau_{(-\infty,0]}) \,|\, S_y(\tau_{(-\infty,0]}) = 0] \\ (S_y^\searrow(n),\, n \ge \tau^\searrow_{(-\infty,0]}) &\equiv 0\,,\end{aligned}$$

where $\tau^\searrow_{(-\infty,0]} := \inf\{n : S_n^\searrow \le 0\}$ and $\tau_{(-\infty,0]} := \inf\{n : S_n \le 0\}$. It has recently been proved in [5], that the rescaled linear interpolation of $S_y^\searrow$, i.e.

$$(n^{-1/\alpha} S^\searrow_{[n^{1/\alpha}x]}([nt]),\, t \ge 0)\,,$$

converges in law on the Skorohod's space as n tends to ∞ towards a Markov process which we will call here the Lévy process (X, P_x) conditioned to hit 0 continuously.

Again, this process may be defined more formally as an h-process of the killed process $(X, \mathbb{P}_x)$ introduced in section 3.1. In this case, the positive harmonic function related to $(X, \mathbb{P}_x)$ is $g(x) = x^{\alpha\rho-1}$, and for $x > 0$, the law $\mathbb{P}_x^{\searrow}$ of the conditioned process is defined by

$$\begin{aligned} &\mathbb{P}_x^{\searrow}(A, t < S) = g(x)^{-1} E_x(g(X_t) \mathbb{I}_A \mathbb{I}_{\{t<T\}}) \\ &\mathbb{P}_x^{\searrow}(X_t = 0, \text{ for all } t \geq S) = 1\,, \end{aligned} \tag{3.13}$$

for all $x > 0$, $t \geq 0$, and $A \in \mathcal{F}_t$. It is proved in [4] that the process $(X, \mathbb{P}_x^{\searrow})$ reaches 0 continuously (it may happen by an accumulation of negative jumps if (X, P_x) has negative jumps), that is

$$\mathbb{P}_x^{\searrow}(X_{S-} = 0) = 1\,,$$

hence $(X, \mathbb{P}_x^{\searrow})$ is a pssMp which belongs to the class $\mathcal{C}_2$. The infinitesimal generator of $(X, \mathbb{P}_x^{\searrow})$ is

$$\mathcal{K}^{\searrow} f(x) = \frac{1}{g(x)} \mathcal{K}(gf)(x)\,, \quad x > 0\,, \ \ f \in \mathfrak{D}_{\mathcal{K}^{\searrow}}\,. \tag{3.14}$$

Trivially, when there are no negative jumps (i.e. $\alpha\rho = 1$), $g \equiv 1$ and the processes $(X, \mathbb{P}_x)$ and $(X, \mathbb{P}_x^{\searrow})$ are the same. The same calculations as in the subsection 3.2, replacing $\alpha\rho$ by $\alpha\rho - 1$ lead to

$$\begin{aligned} \mathcal{K}^{\searrow} f(x) \;&=\; \frac{1}{x^\alpha} \int_{\mathbb{R}^+} [f(ux) - f(x) - x f'(x)(u-1)\mathbb{I}_{\{|u-1|<1\}}] \nu^{\searrow}(u) du \\ &\quad + (a + a_3) x^{1-\alpha} f'(x) + (a(\alpha\rho - 1) + a_4 - k) x^{-\alpha} f(x)\,, \end{aligned}$$

where

$$\begin{aligned} \nu^{\searrow}(u) \;&=\; u^{\alpha\rho-1} \nu(u-1) \\ a_3 \;&=\; c_+ \int_0^1 \frac{(1+x)^{\alpha\rho-1} - 1}{x^\alpha} dx + c_- \int_0^1 \frac{(1-x)^{\alpha\rho-1} - 1}{x^\alpha} dx && (3.15) \\ a_4 \;&=\; c_+ \left(\int_0^1 \frac{(1+x)^{\alpha\rho-1} - 1 - (\alpha\rho - 1)x}{x^{\alpha+1}} dx + \int_1^\infty \frac{(1+x)^{\alpha\rho-1} - 1}{x^{\alpha+1}} dx \right) \\ &\quad + c_- \int_0^1 \frac{(1-x)^{\alpha\rho-1} - 1 + (\alpha\rho - 1)x}{x^{\alpha+1}} dx\,. && (3.16) \end{aligned}$$

Here again, since $(X, \mathbb{P}_x^{\searrow})$ belongs to $\mathcal{C}_2$, the killing rate $a\alpha\rho + a_4 - k$ of its generator must be zero, which gives the following expression for the constant k:

$$k = a(\alpha\rho - 1) + a_4\,. \tag{3.17}$$

Comparing (3.12) with (3.17) we should be able to check that

$$\begin{aligned} &a_4 - a_2 = a = \frac{c_+ - c_-}{1 - \alpha} = \\ &c_+ \left(\int_0^1 \frac{1 - (1+x)^{\alpha\rho-1}}{x^\alpha} dx - \int_1^\infty \frac{(1+x)^{\alpha\rho-1}}{x^\alpha} dx \right) + c_- \int_0^1 \frac{(1-x)^{\alpha\rho-1} - 1}{x^\alpha} dx\,, \end{aligned}$$

however, this seems to be possible to realise only in the trivial cases $\alpha\rho = 1$ and $\alpha = 1$, $\rho = 1/2$.

As in the previous sections, we may now compute the characteristics of the underlying Lévy process in the Lamperti representation of $(X, \mathbb{P}_x^{\searrow})$.

Corollary 3. *Let $\xi^{\searrow}$ be the Lévy process in the Lamperti representation (1.2) of the pssMp $(X, \mathbb{P}_x^{\searrow})$ which is defined in (3.13). The infinitesimal generator $\mathcal{L}^{\searrow}$ of $\xi^{\searrow}$ with domain $\mathfrak{D}_{\mathcal{L}^{\searrow}}$ is given by*

$$\mathcal{L}^{\searrow} f(x) = a^{\searrow} f'(x) + \int_{\mathbb{R}} (f(x+y) - f(x) - f'(x)(e^y - 1)\mathbb{1}_{\{|e^y-1|<1\}})\pi^{\searrow}(y)dy\,,$$

for any $f \in \mathfrak{D}_{\mathcal{L}^{\searrow}}$ and $x \in \mathbb{R}$, where $\pi^{\searrow}(y) = e^{\alpha\rho y}\nu(e^y - 1)$, $y \in \mathbb{R}$ and $a^{\searrow} = a + a_3$, the constant a_3 being defined in (3.15). Equivalently, the characteristic exponent of $\xi^{\searrow}$ is given by

$$\Phi^{\searrow}(\lambda) = ia^{\searrow}\lambda + \int_{\mathbb{R}} (e^{i\lambda y} - 1 - i\lambda(e^y - 1)\mathbb{1}_{\{|e^y-1|<1\}})\pi^{\searrow}(y)dy\,.$$

As we already noticed, the process $(X, \mathbb{P}_x^{\searrow})$ belongs to the class $\mathcal{C}_2$, therefore the underlying Lévy process $\xi^{\searrow}$ drifts to $-\infty$, in particular, since this process is also integrable, we have $-\infty < \mathbb{E}(\xi_1^{\searrow}) = -i\Phi^{\searrow}(0) < 0$.

4 The minimum of ξ up to an independent exponential time.

With the same notations for $(X, \mathbb{P}_x)$, (X, P_x), and ξ as in section 3.1, we suppose here that (X, P_x) has negative jumps, that is $\alpha\rho < 1$ (which is also equivalent to $c_- > 0$). Recall that the characteristics of ξ have been computed in Corollary 1. The first result of this section consists in computing an explicit form of the law of the overall minimum of ξ. Since ξ has finite lifetime, it has the same law as a Lévy process, say $\tilde{\xi}$, with infinite lifetime and killed at an independent exponential time with parameter k. Then let us show how Lamperti representation together with classical results on undershoots of subordinators allow us to compute the law of the minimum of $\tilde{\xi}$ up to an independent exponential time with parameter k. The latter is known as the spacial Wiener-Hopf factor of the Lévy process $\tilde{\xi}$, see [7].

Set $\underline{X} = \inf_{s \le S} X_s$ and $\underline{\xi} = \inf_{s \le \zeta} \xi_s$, where we recall from the introduction that $S = \inf\{t : X_t = 0\}$ and $\zeta := \zeta(\xi)$ is the lifetime of ξ. Then on the one hand, from the Lamperti representation (1.2), under $\mathbb{P}_x$, the processes $\underline{X}$ and $\underline{\xi}$ are related as follows:

$$\underline{X} = x \exp \underline{\xi}\,, \quad \mathbb{P}_x\,\text{–a.s.} \tag{4.1}$$

On the other hand, let H be the downward ladder height process associated to (X, P_0), that is $H_t = -X_{\eta_t}$, where η is the right continuous inverse of the local time

at 0 of the process, (X, P_0) reflected at its minimum, i.e. $(X - \underline{X}, P_0)$. We refer to [1], Chap. VI, for a definition of ladder height processes. It is easy to see the following identity:

$$\underline{X} = x - H_{\nu(x)-}\,, \quad \mathbb{P}_x\text{ -a.s.} \tag{4.2}$$

where $\nu(x) = \inf\{t : S_t > x\}$. In other words, $\underline{X}$ corresponds to the so-called undershoot of the subordinator H at level x. Since H is a stable subordinator with index $\alpha\rho$, the law of $\underline{X}$, and hence this of $\underline{\xi}$, can be computed explicitly as shown in the next proposition. In the sequel, $\mathbb{P}$ will be a reference probability measure under which ξ and H have the laws described above.

Proposition 1. *Recall that $\rho = P_0(X_1 \le 0)$ and let ξ be the Lévy process which law is described in Corollary 1. For any $\lambda > 0$,*

$$\mathbb{E}(e^{\lambda\underline{\xi}}) = \frac{\Gamma(\lambda + 1 - \alpha\rho)}{\Gamma(\lambda+1)\Gamma(1-\alpha\rho)}\,. \tag{4.3}$$

In other words, $\exp\underline{\xi}$ is a Beta variable with parameters $\alpha\rho$ and $1-\alpha\rho$, i.e. $\exp\underline{\xi}$ has density function: $\mathbb{P}(\exp\underline{\xi} \in dt) = \beta(\alpha\rho, 1-\alpha\rho)^{-1} t^{\alpha\rho-1}(1-t)^{-\alpha\rho}\mathbb{1}_{\{t\in[0,1]\}}\,dt$.

Proof. Recall that the Lévy measure $\theta(dy)$ of H and its potential measure $U(dy)$ are given by:

$$\theta(dy) = c_1 y^{-(\alpha\rho+1)}\mathbf{1}_{\{y>0\}}\,dy \quad \text{and} \qquad \int_0^\infty e^{-\lambda y}\,U(dy) = c_2\lambda^{-\alpha\rho}\,,$$

where c_1 and c_2 are positive constants. Then from Proposition 2 of [1], Chap. III,

$$\mathbb{P}(H_{\nu(x)-} \in dy) = \mathbb{1}_{\{y\in[0,x]\}}\int_x^\infty U(dy)\theta(dz-y)\,,$$

from which we obtain for all $\lambda \ge 0$ and $\mu \ge 0$,

$$\begin{aligned}\int_0^\infty e^{-\mu x}E(e^{-\lambda(x-H_{\nu(x)-})})\,dx &= \int_0^\infty e^{-(\lambda+\mu)x}\int_0^x e^{\lambda y}\int_x^\infty U(dy)\,\theta(dz-y)\,dx\\ &= \int_0^\infty e^{-(\lambda+\mu)x}\int_0^x \frac{c_1}{\alpha\rho}e^{\lambda y}(x-y)^{-\alpha\rho}U(dy)\,dx\\ &= \frac{c_1c_2(\lambda+\mu)^{\alpha\rho-1}}{\alpha\rho\mu^{\alpha\rho}}\Gamma(1-\alpha\rho) = \frac{(\lambda+\mu)^{\alpha\rho-1}}{\mu^{\alpha\rho}}\,.\end{aligned}$$

It means that if ς is exponentially distributed with parameter μ and independent of H, then $\varsigma - H_{\nu(\varsigma)-}$ is gamma distributed with parameters μ and $1-\alpha\rho$, i.e.

$$\mathbb{E}\left(e^{-\lambda(\varsigma - H_{\nu(\varsigma)-})}\right) = \left(\frac{\mu}{\lambda+\mu}\right)^{1-\alpha\rho}\,.$$

Recall that the moment of order $\lambda > 0$ of the Gamma law with parameters μ and $1-\alpha\rho$ is $\Gamma(\lambda+1-\alpha\rho)/(\mu^\lambda\Gamma(1-\alpha\rho))$, then thanks to (4.1) and (4.2), one has

$$\mathbb{E}(e^{\lambda\underline{\xi}}) = \frac{\mathbb{E}(\gamma^\lambda)}{\mathbb{E}(\varsigma^\lambda)} = \frac{\Gamma(\lambda+1-\alpha\rho)}{\Gamma(\lambda+1)\Gamma(1-\alpha\rho)},$$

which is the moment of order λ of a Beta variable with parameters $\alpha\rho$ and $1-\alpha\rho$. $\square$

In view of the result of Proposition 1, one is tempted to compute the law of the overall minimum $\inf_{t\le\mathbf{e}(\mu)}\tilde{\xi}_t$ of the unkilled process $\tilde{\xi}$ before an independent exponential time of *any* parameter $\mu > 0$. However although the pssMp which is obtained from $(\tilde{\xi}_t,\, t \le \mathbf{e}(\mu))$ through Lamperti representation is absolutely continuous with respect to $(X, \mathbb{P}_x)$, its law is not sufficiently explicit to apply the same arguments a in Proposition 1.

We can still apply the same arguments as above to determine to law of the overall minimum of the Lévy process $\xi^\uparrow$ which is defined in section 3.2. Indeed, as we observed in this section, $\xi^\uparrow$ drifts to $+\infty$, as well as the pssMp $(X, \mathbb{P}_x^\uparrow)$, and from Lamperti representation the relation

$$\underline{X} = x\exp\underline{\xi}^\uparrow, \quad \mathbb{P}_x^\uparrow\text{-a.s.} \tag{4.4}$$

holds. Moreover, the law of $(\underline{X}, \mathbb{P}_x^\uparrow)$ is explicit and may be found in [4], Theorem 5: for all $x > 0$,

$$\mathbb{P}_x^\uparrow(\underline{X} \le y) = \frac{x^{\alpha\rho} - (x-y)^{\alpha\rho}\mathbb{I}_{\{y\le x\}}}{x^{\alpha\rho}}.$$

This allows us to state:

Proposition 2. *Let $\xi^\uparrow$ be the Lévy process which law is described in Corollary 2. The law of the overall minimum $\underline{\xi}^\uparrow$ of $\xi^\uparrow$ is given by:*

$$\mathbb{P}(-\underline{\xi}^\uparrow \le z) = (1-e^{-z})^{\alpha\rho}\mathbb{I}_{\{z\ge 0\}}. \tag{4.5}$$

This computation is closely related to risk theory and in particular proposition 2 provides an explicit form of the ruin probability at level $z \ge 0$, i.e.

$$\mathbb{P}(\exists\, t\ge 0,\, z+\xi_t^\uparrow \le 0) = \mathbb{P}(\underline{\xi}^\uparrow \le -z) = 1-(1-e^{-z})$$

for this class of Lévy processes, see the recent paper by Lewis and Mordecki [11].

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